Background

A data vector that exists as the sum of signal plus systematic noise plus shot noise will be denoted in the following ways for these four spaces,

$$\begin{aligned} \boldsymbol{\Gamma} &= \boldsymbol{\kappa} + \boldsymbol{\eta} + \boldsymbol{\zeta}, && cell\ space \\ \boldsymbol{M} &= \boldsymbol{S} + \boldsymbol{T} + \boldsymbol{Y}, && signal\ space \\ \boldsymbol{m} &= \boldsymbol{s} + \boldsymbol{t} + \boldsymbol{y}, && W\ space \\ \boldsymbol{\xi} &= \boldsymbol{\omega} + \boldsymbol{\phi} + \boldsymbol{\pi}, && B\ space \end{aligned}$$

$\boldsymbol{\Gamma}$, $\boldsymbol{M}$, $\boldsymbol{m}$ and $\boldsymbol{\xi}$ are data vectors; $\boldsymbol{\kappa}$, $\boldsymbol{S}$, $\boldsymbol{s}$ and $\boldsymbol{\omega}$ are signal vectors; $\boldsymbol{\eta}$, $\mathbf{T}$, $\boldsymbol{t}$ and $\boldsymbol{\phi}$ are systematic noise vectors; and $\boldsymbol{\zeta}$, Y, y and $\boldsymbol{\pi}$ are shot noise vectors in their respective spaces.

The signal covariance matrix is

$$\boldsymbol{\Sigma}_{\kappa} = \mathbf{Z}\boldsymbol{\Lambda}^{(\kappa)}\mathbf{Z}^{T},$$

where the i^{th} column of the orthonormal matrix $\mathbf{Z}$ contains the i^{th} eigenvector $\hat{\boldsymbol{z}}_i$ of $\boldsymbol{\Sigma}_{\kappa}$ and the i^{th} diagonal element of $\boldsymbol{\Lambda}^{(\kappa)}$ equals $\boldsymbol{\Sigma}_{\kappa}$'s i^{th} eigenvalue $\lambda_i^{(\kappa)}$.

$$\begin{aligned} \boldsymbol{S} &= \mathbf{Z}^{T}\boldsymbol{\kappa}, \\ \boldsymbol{\kappa} &= \mathbf{Z}\boldsymbol{S}. \end{aligned}$$

As an orthonormal matrix, the columns of $\mathbf{Z}$ comprise a full basis that spans cell-space. Therefore any signal vector $\boldsymbol{\kappa}$ can be represented as a linear combination of them,

$$\boldsymbol{\kappa} = \sum_{i=1}^{N_{cells}} S_i \hat{\boldsymbol{z}}_i,$$

where the expansion coefficient S_i equals the projection of the signal onto the i^{th} signal mode, $S_i = \boldsymbol{\kappa} \cdot \hat{\boldsymbol{z}}_i$. In the basis $\mathbf{Z}$, the signal coefficients are mean zero, $\langle S_i \rangle = 0$, and statistically orthogonal,

$$\langle S_i {S_j}^{*} \rangle = \begin{cases} \lambda_j^{(\kappa)} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Analytic Power Spectrum – General Construction

Using Signal Eigenmodes

Let's start by figuring out the power spectrum in discretized bins. There are a couple different ways to do this, but I will employ a non-parametric estimator that utilizes the signal eigenmodes. I establish in Appendix A that the power spectrum of the clustering signal can be represented as a variance weighted sum over all N Fourier-transformed signal eigenvectors $\hat{\boldsymbol{z}}$,

$$P_\kappa(\boldsymbol{k}) = \langle|\kappa(\boldsymbol{k})|^2\rangle = \sum_{m=1}^{N} \lambda_m^{(\kappa)} |\hat{\boldsymbol{z}}_m(\boldsymbol{k})|^2.$$

Because the eigenvectors $\hat{\boldsymbol{z}}$ are discretized, the best we can do in finding each $\hat{\boldsymbol{z}}_m(\boldsymbol{k})$ is a fast Fourier transform. This means we will have to split k-space into a discrete number of (about 480 or so) bins. Let the boundaries of the n^{th} bin be k_n and k_{n+1}. The power in this bin is the average of the Fourier amplitudes between those two band powers,

$$P_{\kappa,n} = \sum_{m=1}^{N} \lambda_m^{(\kappa)} \langle|\hat{\boldsymbol{z}}_m(\boldsymbol{k})|^2\rangle_n,$$

where

$$\langle|\hat{\boldsymbol{z}}_m(\boldsymbol{k})|^2\rangle_n = \begin{cases} \langle|\hat{\boldsymbol{z}}_m(\boldsymbol{k})|^2\rangle & k_n \le |\boldsymbol{k}| < k_{n+1} \\ 0 & \text{otherwise} \end{cases}.$$

Using the eigenvalues $\lambda_m^{(\kappa)}$ will recover the fiducial power spectrum used in our model. But to assess the quality of the signal reconstruction, we should replace this with the variance of the mean-zero estimated signal coefficients in signal-space such that

$$P_{\kappa,n} = \sum_{m=1}^{N} \hat{S}_m^2 B_{mn},$$

where $B_{mn} \equiv \langle|\hat{\boldsymbol{z}}_m(\boldsymbol{k})|^2\rangle_n$ and $\hat{\boldsymbol{s}} = \mathbf{Z}^T\hat{\boldsymbol{\kappa}}$. Note that once the signal modes and Fourier grid spacings are set, the B coefficients need only be solved for once. New estimated signal coefficients can be plugged in thereafter to yield their unique power spectrum.

The analytical method finds the power spectra using the variances of various signal and noise components. To plot their spectra these variances must be computed in signal-space.

SIGNAL

The variance of the signal along each dimension of signal-space is trivial,

$$\mathrm{Var}(S_m) = \lambda_m^{(\kappa)}.$$

SHOT NOISE

In signal-space, a shot noise overdensity vector is $\vec{Y} = \mathbf{Z}^T\vec{\zeta}$. The m^{th} element of $\vec{Y}$ equals $Y_m = \sum_{i=1}^{N} \mathrm{Z}_{i,m}\zeta_i$. The variance of this component is

$$\mathrm{Var}(Y_m) = \sum_{i=1}^{N} \mathrm{Z}_{i,m}^2\,\mathrm{Var}(\zeta_i) = \sum_{i=1}^{N} \mathrm{Z}_{i,m}^2 \frac{1}{\langle n_i\rangle},$$

where $\langle n_i\rangle$ is the expected number of galaxies in the i^{th} cell.

ZERO-POINT NOISE

In signal-space, a zero-point noise overdensity vector is $\vec{T} = \mathbf{Z}^T\vec{\eta}$ while in noise-space we have $\vec{\eta} = \mathbf{U}\,\mathfrak{d}_\eta$. It follows that $\vec{T} = \mathbf{R}\mathfrak{d}_\eta$ where $\mathbf{R} = \mathbf{Z}^T\mathbf{U}$. The m^{th} element of $\vec{T}$ equals $T_m = \sum_{i=1}^{N} \mathrm{R}_{m,i}\mathfrak{d}_\eta[i]$. The variance of this component is

$$\mathrm{Var}(T_m) = \sum_{i=1}^{N} \mathrm{R}_{m,i}^2\,\mathrm{Var}\big(\mathfrak{d}_\eta[i]\big) = \sum_{i=1}^{N} \mathrm{R}_{m,i}^2\lambda_i^{(\eta)}.$$

DATA VECTOR

Because signal, shot noise and zero-point noise are independent of one another, it follows that

$$\mathrm{Var}\big(\vec{M}\big) = \mathrm{Var}\big(\vec{S} + \vec{T} + \vec{Y}\big) = \mathrm{Var}\big(\vec{S}\big) + \mathrm{Var}\big(\vec{T}\big) + \mathrm{Var}\big(\vec{Y}\big)$$

$$\mathrm{Var}(M_m) = \lambda_m^{(\kappa)} + \sum_{i=1}^{N} \mathrm{M}_{m,i}^2\lambda_i^{(\eta)} + \sum_{i=1}^{N} \mathrm{Z}_{i,m}^2 \frac{1}{\langle n_i\rangle}.$$

Each dimension's variance subsequently scales the B_{mn} coefficients which are then combined to yield power spectra.

We can repeat the analysis by using the Fourier transforms of the eigenmodes in cell-space. This situation offers the advantage of not needing to recreate Fourier modes and B_{mn} coefficients if the signal model changes. The eigenmodes are also straightforward in that each is a unit vector with value 1 in one element and 0 in all others.

We allow the discretized set of cells to act as a standard basis where unit vector $\hat{\boldsymbol{e}}_m$ equals 1 in the m^{th} element and zero otherwise. As an example let's consider the power of the shot noise since this is diagonal in cell-space.

$$P_\zeta(\boldsymbol{k}) = \langle|\zeta(\boldsymbol{k})|^2\rangle = \sum_{m=1}^{N} \frac{1}{\langle n_m\rangle}|\hat{\boldsymbol{e}}_m(\boldsymbol{k})|^2.$$

Because the eigenvectors $\hat{\boldsymbol{e}}$ are discretized, the best we can do in finding each $\hat{\boldsymbol{e}}_m(\boldsymbol{k})$ is a fast Fourier transform. This means we will have to split k-space into a discrete number of (about 480 or so) bins. Let the boundaries of the n^{th} bin be k_n and k_{n+1}. The power in this bin is the average of the Fourier amplitudes between those two band powers,

$$P_{\zeta,n} = \sum_{m=1}^{N} \frac{1}{\langle n_m\rangle}\langle|\hat{\boldsymbol{e}}_m(\boldsymbol{k})|^2\rangle_n,$$

where

$$E_{mn} \equiv \langle|\hat{\boldsymbol{e}}_m(\boldsymbol{k})|^2\rangle_n = \begin{cases}\langle|\hat{\boldsymbol{e}}_m(\boldsymbol{k})|^2\rangle & k_n \le |\boldsymbol{k}| < k_{n+1} \\ 0 & \text{otherwise}\end{cases}.$$

Using the variance $1/\langle n_m\rangle$ in equation X will deliver the power spectrum of the shot noise model. Because the overdensity of the shot noise (and signal and zero-point noise for that matter) equals zero for all m, the variance equals the expected square of the shot noise term. The power spectrum of a shot noise realization is therefore calculable through

$$P_{\zeta,n} = \sum_{m=1}^{N} \zeta_m^2 E_{mn},$$

Rather than find the power spectra of individual vectors, the analytical method finds the power spectra using the variances of various signal and noise components. The standard basis defined by cell-space has already been Fourier transformed and the results stored in the E_{mn} coefficients. What remains is expressing the variances of each component in cell-space. Consider a random data vector in cell-space, $\boldsymbol{\Gamma} = \boldsymbol{\kappa} + \boldsymbol{\eta} + \boldsymbol{\zeta}$.

SHOT NOISE

The variance of the shot noise along each dimension of cell-space is trivial,

$$\mathrm{Var}(\zeta_m) = \frac{1}{\langle n_m \rangle}.$$

SIGNAL

In cell-space a signal vector is $\kappa = \mathbf{Z}\,\mathrm{S}$. The m^{th} element of κ equals $\kappa_m = \sum_{i=1}^{N} \mathrm{Z}_{i,m}\, S_i$. The variance of this component is

$$\mathrm{Var}(\kappa_m) = \sum_{i=1}^{N} \mathrm{Z}_{i,m}^2\, \mathrm{Var}(S_i) = \sum_{i=1}^{N} \mathrm{Z}_{i,m}^2\, \lambda_m^{(\kappa)},$$

ZERO-POINT NOISE

In cell-space a zero-point noise vector is $\eta = \mathbf{U}\,\mathfrak{d}_\eta$. The m^{th} element of η equals $\eta_m = \sum_{i=1}^{N} \mathrm{U}_{i,m}\, \mathfrak{d}_\eta[i]$. The variance of this component is

$$\mathrm{Var}(\eta_m) = \sum_{i=1}^{N} \mathrm{U}_{i,m}^2\, \mathrm{Var}\big(\mathfrak{d}_\eta[i]\big) = \sum_{i=1}^{N} \mathrm{U}_{i,m}^2\, \lambda_m^{(\eta)},$$

DATA VECTOR

Because signal, shot noise and zero-point noise are independent of one another, it follows that

$$\begin{aligned} \mathrm{Var}(\boldsymbol{\Gamma}) &= \mathrm{Var}(\boldsymbol{\kappa} + \boldsymbol{\eta} + \boldsymbol{\zeta}) \\ &= \mathrm{Var}(\boldsymbol{\kappa}) + \mathrm{Var}(\boldsymbol{\eta}) + \mathrm{Var}(\boldsymbol{\zeta}) \end{aligned}$$

$$\mathrm{Var}(\Gamma_m) = \sum_{i=1}^{N} \mathrm{Z}_{i,m}^2\, \lambda_m^{(\kappa)} + \sum_{i=1}^{N} \mathrm{U}_{i,m}^2\, \lambda_m^{(\eta)} + \frac{1}{\langle n_m \rangle}.$$

Each dimension's variance subsequently scales the E_{mn} coefficients which are then combined to yield power spectra.

Fourier Transforming the Modes

This process is too complicated to detail here in total. I'll cover the bullet points.

- Construct a 512^3 FFT grid that is more than twice the length of the survey in each direction. In this way the vast majority of gridboxes will equal zero.

- Split each eigenvector element fractionally amongst the grid boxes its cell intersects.

- Use MATLAB's `fftn` routine to Fourier transform the vector created in the above step. The output looks something like this:

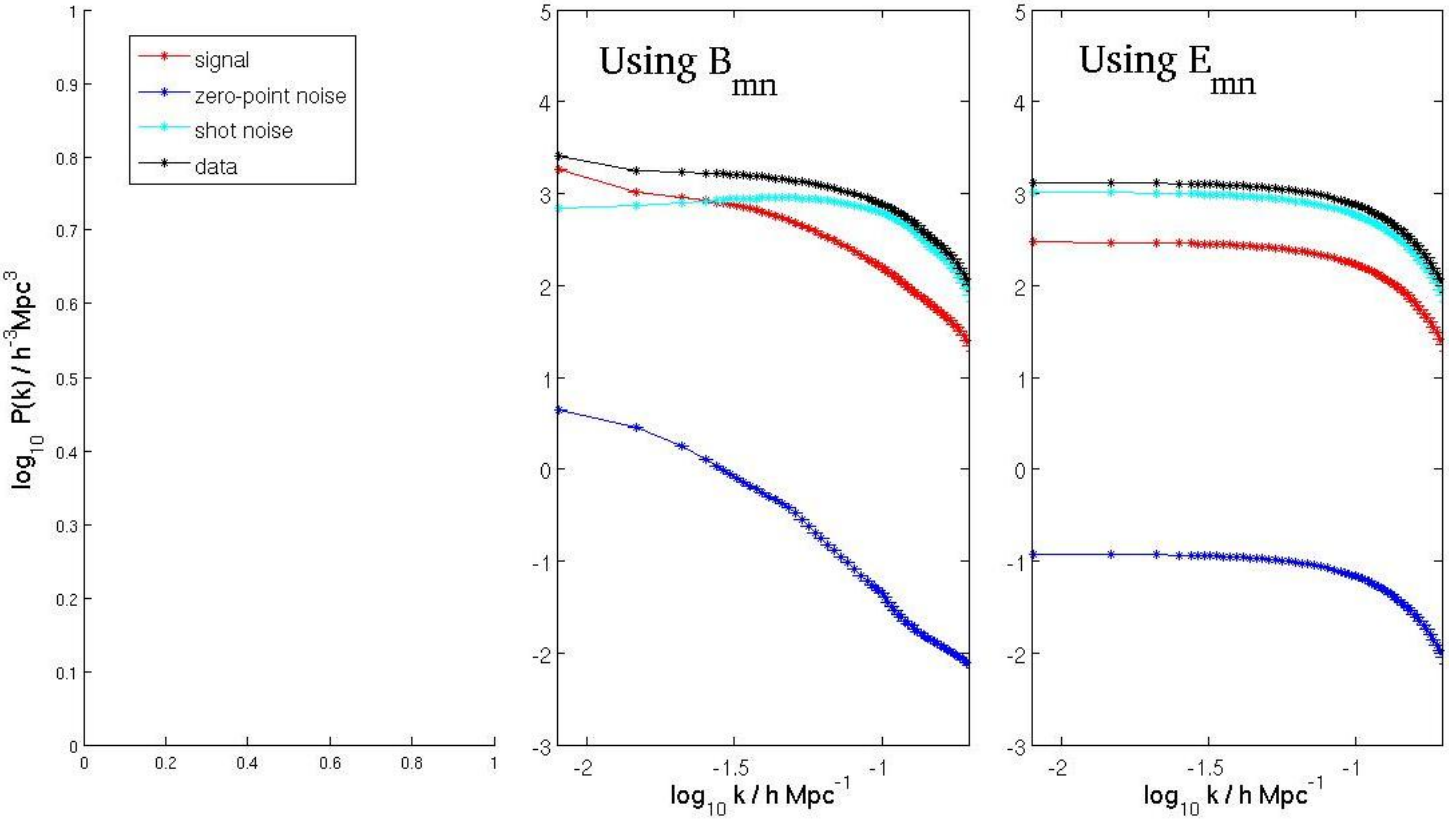
- The following subroutine breaks the signal into grid boxes, Fourier transforms it and exports the coefficients as they would be reported in C.

C index	n_x	n_y	n_z	$\|\boldsymbol{k}\| = kg(:,4)$	sd	zk
1	0	0	0	$\boldsymbol{k}_1$	1	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_1)\|^2$
2	0	0	1	$\boldsymbol{k}_2$	2	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_2)\|^2$
3	0	0	2	$\boldsymbol{k}_3$	2	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_3)\|^2$
4	0	0	3	$\boldsymbol{k}_4$	2	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_4)\|^2$
5	0	0	4	$\boldsymbol{k}_5$	1	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_5)\|^2$
6	0	1	0	$\boldsymbol{k}_6$	1	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_6)\|^2$
7	0	1	1	$\boldsymbol{k}_7$	2	$\|\hat{\boldsymbol{z}}(\boldsymbol{k}_7)\|^2$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n^2(n/2+1)$	-1	-1	$\pm n/2$	$\boldsymbol{k}_{n^2(n/2+1)}$	1	$\left\|\hat{\boldsymbol{z}}\left(\boldsymbol{k}_{n^2(n/2+1)}\right)\right\|^2$

- The $B_{mn} = \langle|\hat{\boldsymbol{z}}_m(\boldsymbol{k})|^2\rangle_n$ coefficients are averaged using the above table and used to scale the variances in each bin like $P_{\kappa,n} = \sum_{m=1}^{N} \hat{S}_m^2 B_{mn}$.

A Discrepancy?

I tried using both the B_{mn} and E_{mn} coefficients to plot the analytic spectra. I see no reason why they shouldn't yield the same results, but here's what comes out.



This leads me to suspect that the method of developing the coefficients is incorrect.

Analytic Power Spectrum – Fiducial Power Spectrum

We start with the fiducial power spectrum convolved with top-hat window functions. The most appropriate filtering kernel, or window W_R, is the following,

$$K(\boldsymbol{r}) = K(r) = \begin{cases} 3/4\pi R^3 & \text{if } r \le R \\ 0 & \text{if } r > R \end{cases}.$$

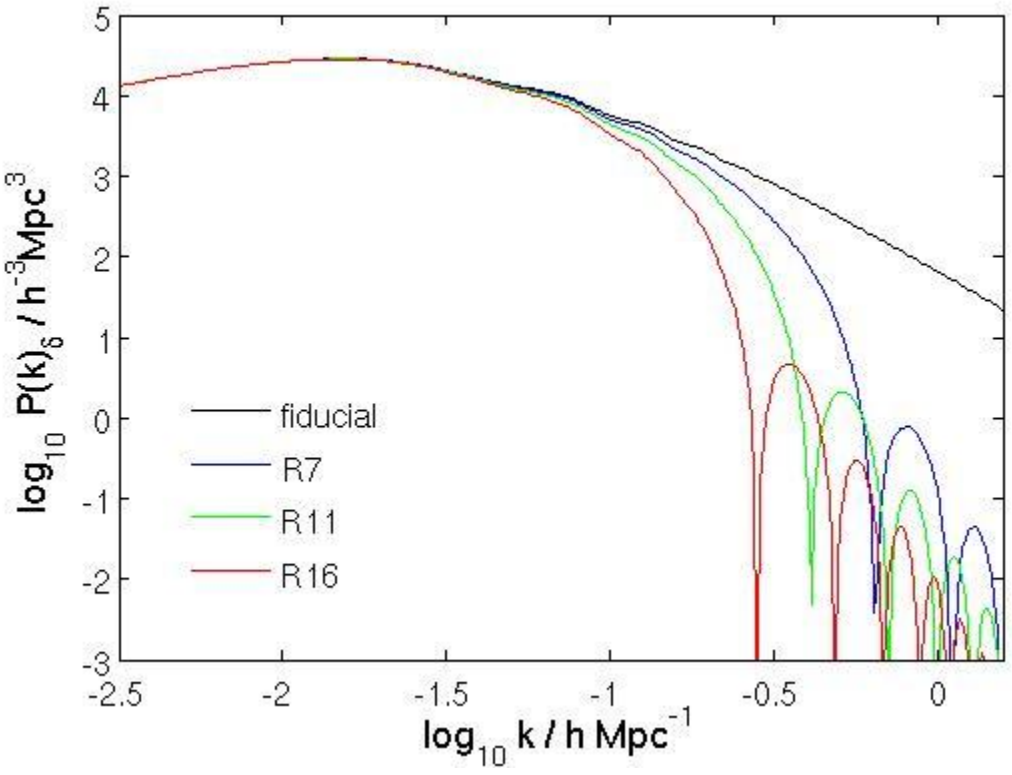
In Fourier space the window function is

$$|W_R(k)|^2 = \left| \int K(\boldsymbol{r})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3r \right|^2 = \left| 3\,\frac{j_1(kR)}{kR} \right|^2.$$

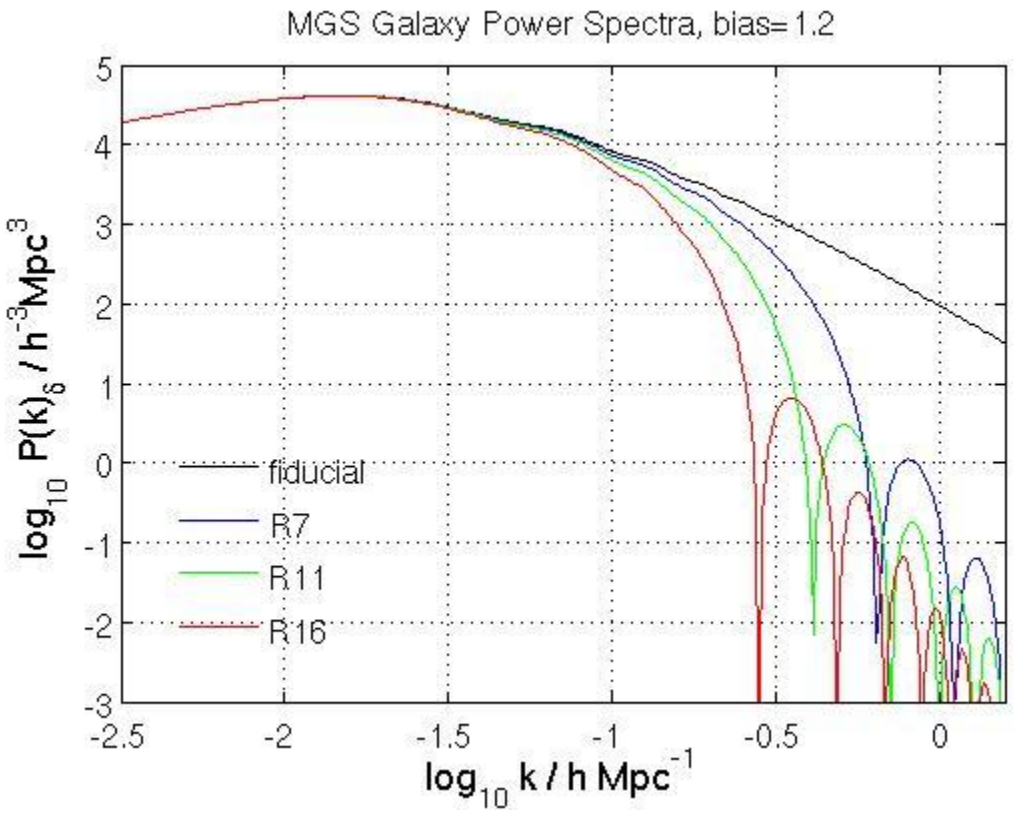
The convolved power spectrum is

$$P(k) = |W_R(k)|^2 P_{fid}(k).$$

Here are the matter power spectra.



Here are the galaxy power spectra assuming $b = 1.2$.



SML98 approximate the redshift space correlation function to be

$$\xi^{(s)}(r,\theta,\gamma) = c_{00}\xi_0^{(0)} + c_{02}\xi_2^{(0)} + c_{04}\xi_4^{(0)}.$$

where

$$\xi_L^{(n)}(r) = \frac{1}{2\pi^2}\int dk\, k^2\, j_L(kr)P(k).$$

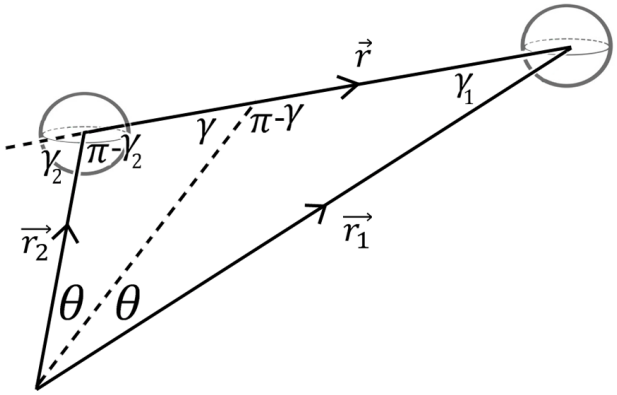
Here j_L is the spherical Bessel function. P_l is the Legendre polynomial. The coefficients are

$$c_{00} = 1 + \frac{2}{3}\beta + \frac{1}{5}\beta^2 - \frac{4}{15}\beta^2\cos^2\theta\,\sin^2\theta$$

$$c_{02} = -\left(\frac{4}{3}\beta + \frac{4}{7}\beta^2\right)\cos 2\theta\ P_2(\cos\gamma) - \frac{2}{3}\left(\beta - \frac{1}{7}\beta^2 + \frac{4}{7}\beta^2\sin^2\theta\right)\sin^2\theta$$

$$c_{04} = \frac{8}{35}\beta^2 P_4(\cos\gamma) - \frac{4}{21}\beta^2\sin^2\theta\, P_2(\cos\gamma) - \frac{1}{5}\beta^2\left(\frac{4}{21} - \frac{3}{7}\sin^2\theta\right)\sin^2\theta.$$

The angles follow this geometry,



$$2\theta = \cos^{-1}\left(\frac{\mathbf{r_1} \cdot \mathbf{r_2}}{r_1 r_2}\right),$$

$$r^2 = {r_1}^2 + {r_2}^2 - 2r_1 r_2 \cos(2\theta)\,,$$

$$\gamma = \sin^{-1}\left(\frac{r_1 + r_2}{r}\sin\theta\right).$$

There is an issue regarding which value of r to use. The Euclidean distance between the centers of the cells is the easy choice, but there's also the Liske geometry which takes time lag effects into consideration. This geometry is explained in Appendix B.

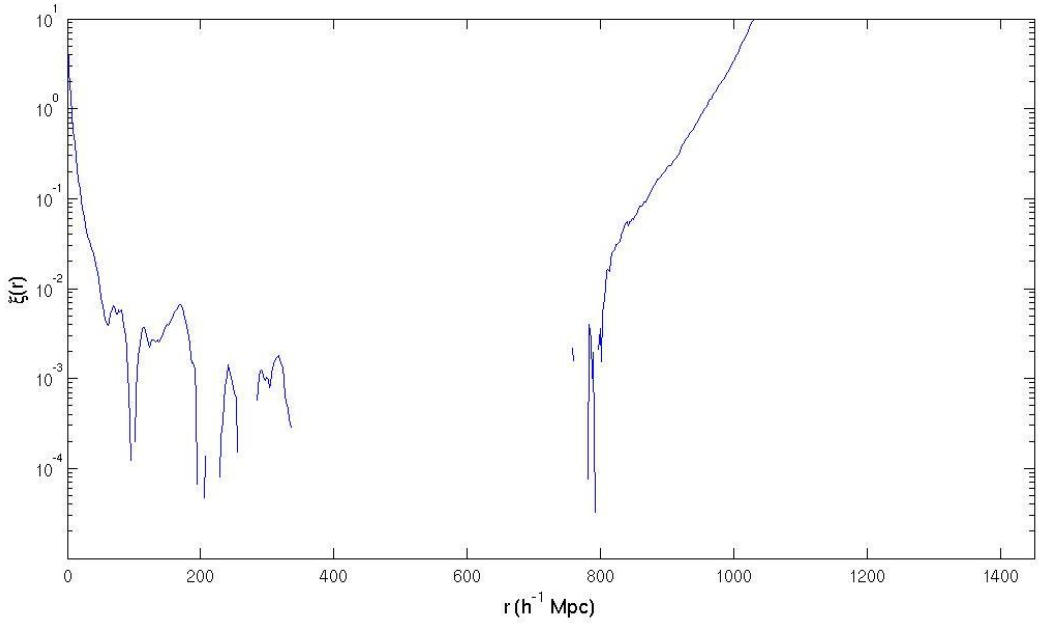
The distances and angles between the cells defines which value of $\xi^{(s)}(r, \theta, \gamma)$ to place in the appropriate element of the signal covariance matrix (in cell-space) $\mathbf{\Sigma}_{\kappa}$.

Analytic Power Spectrum – Empirical 2PCF

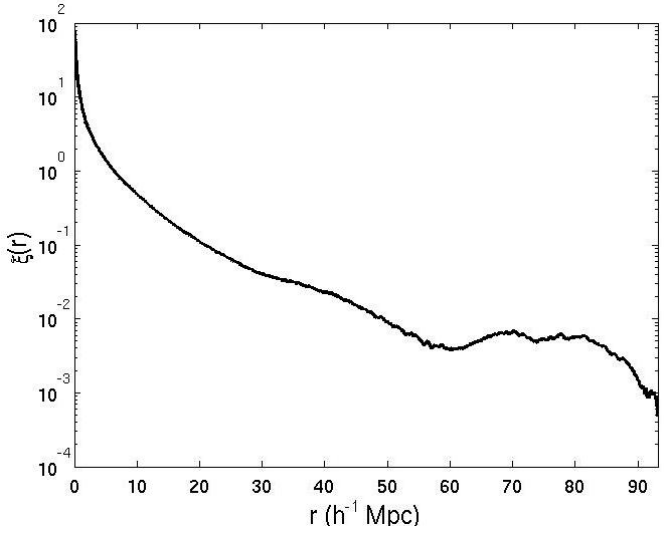
I consider all $N = 441{,}308$ spectroscopically observed MGS targets within the northern STRIPEs of the improved spectroscopic footprint. I calculate the 2PCF using $(DD - 2DR + RR)/RR$. I count the number of galaxy pairs at each radial separation r (defined in a Euclidean sense) and normalize those counts in bins by dividing by $N(N-1)/2$.

I do the same for $N_R = 10N$ random points. The number of distinct RR pairs is $N_R(N_R - 1)/2$. To correct for edge effects I also need DR pair counts of which there will be NN_R. The result is a two-column table. The resulting function is pictured below.

$r, \xi(r)$	`base/TwoPCF_MGS_pristine_z0p02_0p30_b0p05_32kRows_fwriteR`



The correlation function first runs negative at $r_n = 93.175\ h^{-1}$ Mpc. I set $\xi(r) = 0$ for $r > r_n$. I execute a moving average with a 12-element window to smooth out much of the noise. Each element is $1/12$ and it's convolved with $\xi(r)$. Here's the result,



There are actually two different ways this was done. In the first, the randoms were mistakenly constrained to $0.02 \leq z \leq 0.22$ instead of extending to $z = 0.30$. Only MGS targets with spectra were considered. In the second, the randoms were extended to $z = 0.30$ and the lost objects were assigned the redshifts of their nearest neighbors.

I convolve these with the window functions by translating into Fourier space and back again.

$$P(k) = 4\pi \int dr\ r^2\ \frac{sin(kr)}{kr} \xi(r).$$

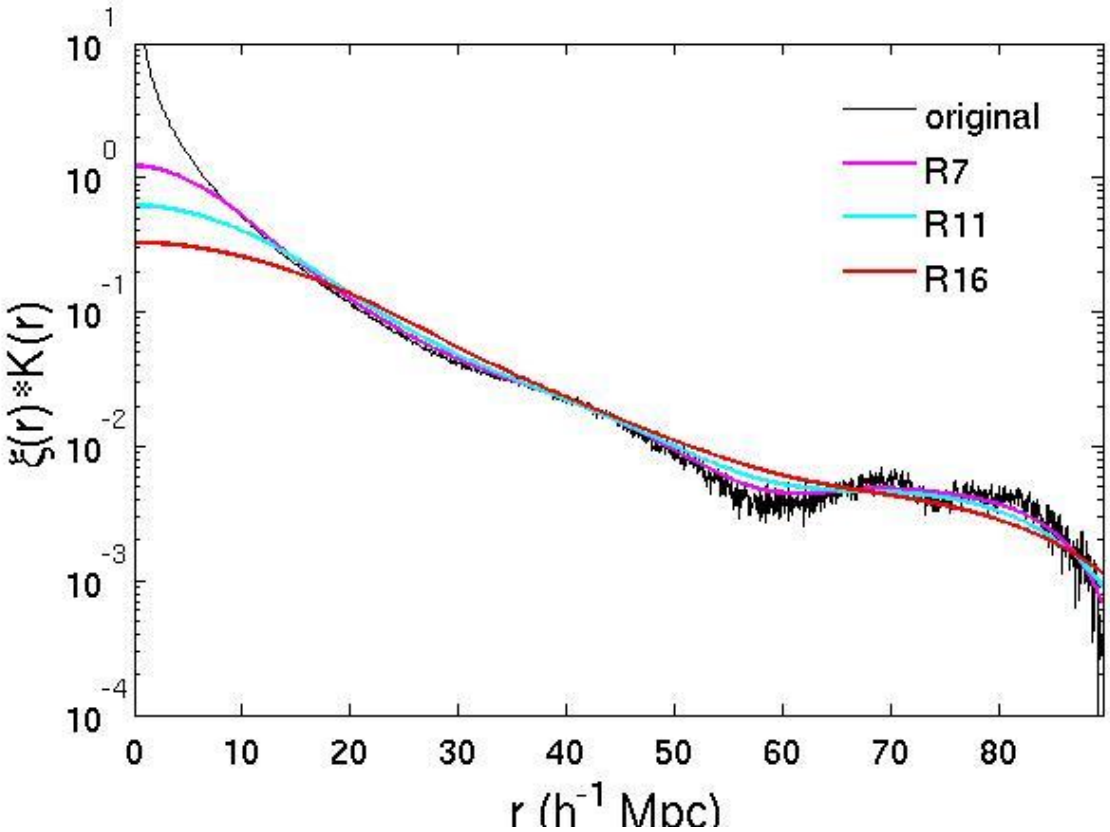
The convolved power spectrum is

$$P(k,R) = \left| 3\ \frac{j_1(kR)}{kR} \right|^2 P(k).$$

Then translate back.

$$\xi(r) * K(r) = \frac{1}{2\pi^2} \int dk\ \ k^2 \frac{\sin(kr)}{kr} P(k,R).$$

Both integrations occur numerically with $\Delta r = 0.01$ from $r = 0$ to $r = 100$ and $\Delta k = 0.0025$ from $k = 0.0025$ to $k = 100$ (though it should have been 32…the fix didn't seem to change things). It's possible that going back and forth between configuration and Fourier spaces introduces some numerical problems, but the figures below seem to suggest that the convolution is happening more or less correctly.



[caption: 2-point correlation functions determined empirically from MGS galaxies. Random points extended to $z = 0.30$. Lost objects were assigned redshifts of their nearest neighbors.]

Appendix A

We know that each element of the noise modes refers to a particular cell in space, so I should be able to take the Fourier transform of the noise modes in much the same way as I took the Fourier transform of the number counts. This would include addressing the window function and normalization as before. In this way I can generate a graph of the power spectrum of the noise modes themselves, a good figure to get no matter what.

One can find the Fourier transform of the n^{th} noise mode, $\psi_n(k)$, weight them using the eigenvalue of the n^{th} mode, $\lambda_n^{(\psi)}$, and sum them like this

$$\sum \lambda_n^{(\psi)} \left|\hat{\psi}_n(k)\right|^2.$$

To see why, note that I can expand any particular fluctuation due to the zero-point error, f_i, where i is the cell label.

$$\delta_i = \sum_n a_n \psi_n(i),$$

where a_n is an unknown but random number. The above equation is a decomposition of the noise fluctuations into the noise modes. The Fourier transform of this would be

$$\hat{\delta}(k) = \sum_n a_n \hat{\psi}_n(k).$$

To get the power spectrum of this I would need to square it and take the absolute value,

$$\langle |\delta(k)|^2 \rangle = \sum_{m,n} \langle a_m a_n \rangle \hat{\psi}_m(k) \hat{\psi}_n{}^*(k).$$

Because the noise modes are orthogonal, these should be as well, giving this a diagonal quality. Further, a_m and a_n should be uncorrelated. a_n represents the overlap of one's noise vector with ψ_n, or $a_n = \delta \cdot \psi_n$. To find $\langle a_m a_n \rangle$ I begin with the matrix

$$\langle \delta \delta^T \rangle = \operatorname{Cov} \delta = A \Lambda A^T,$$

where

$$A = \begin{bmatrix} | & | & & | \\ \psi_1 & \psi_2 & \cdots & \psi_n \\ | & | & & | \end{bmatrix}.$$

I note that if $a = [a_1 \quad a_2 \quad \cdots \quad a_n]^T$, then δ may be expanded as $\delta = Aa$. Then

$$\langle \delta \delta^T \rangle = \langle A a a^T A^T \rangle = A \langle a a^T \rangle A^T.$$

A comparison with the above expression reveals $\langle \delta\delta^T \rangle = A\Lambda A^T = A\langle aa^T \rangle A^T$ or

$$\langle aa^T \rangle = \Lambda.$$

The diagonal matrix Λ is filled mostly with zeros such that

$$\langle a_m a_n \rangle = \lambda_n^{(\psi)} \delta_{mn},$$

where δ_{mn} is the Kronecker delta. Putting this all together, you see that the noise power spectrum is predictable directly from the eigenvalues and eigenmodes of the noise,

$$\langle |\delta(k)|^2 \rangle = \sum \lambda_n^{(\psi)} \left|\hat{\psi}_n(k)\right|^2.$$

Appendix B

$\xi(r)$ is a function of the distance between points in space. For two galaxies, its magnitude cannot be measured directly. Instead, it must be calculated based upon the galaxies' redshifts z_1 and z_2 and their angular separation α.

Suppose Galaxy 2 emits a photon towards Galaxy 1. At the instant Galaxy 1 receives the photon, it emits one of its own bound for earth that is later measured at z_1. For the purposes of the correlation function, the relevant causally connected distance is $r = \chi_2'$, the comoving separation between Galaxies 1 and 2 at the moment Galaxy 1 receives and emits its photons.

The solution for χ_2' (and z_2', the redshift of the photon received by Galaxy 1) was worked out by J. Liske (Mon. Not. R. Astron. Soc. **319**, 557-561 (2000)) whose results I summarize here. These derivations assume a homogeneous Friedmann (zero-pressure) cosmology with no cosmological constant ($\Lambda = 0$). Of course this is not strictly true, but since my survey occupies a relatively small volume of redshift space ($z < 0.22$) and correlations between distant volumes are quite small anyhow, this approximation should be acceptable.

The solution below incorporates my continued assumption of a flat universe. Here are some values needed for the final distance calculation.

$$P_+ = \frac{1}{q_0}\left[(q_0 - 1)\left(\sqrt{1 + 2q_0 z_1} + \sqrt{1 + 2q_0 z_2} - 1\right) + \sqrt{(1 + 2q_0 z_1)(1 + 2q_0 z_2)} - q_0\right],$$

$$P_- = \sqrt{1 + 2q_0 z_2} - \sqrt{1 + 2q_0 z_1},$$

$$P^2 = \frac{1}{4}\frac{1 + z_1}{1 + z_2}\frac{1}{1 + 2q_0 z_1}\left({P_+}^2 \sin^2\frac{\alpha}{2} + {P_-}^2 \cos^2\frac{\alpha}{2}\right),$$

$$q_1 = q_0\frac{1 + z_1}{1 + 2q_0 z_1},$$

$$z_2' = \frac{2P^2}{(q_1 - 2P^2)^2}\left(1 + q_1 - 2P^2 + \sqrt{\frac{{q_1}^2}{P^2} + 1 - 2q_1}\right),$$

$$H_1 = H_0(1 + z_1)\sqrt{1 + 2q_0 z_1},$$

$$a_1 = \frac{a_0}{1 + z_1},$$

$$\chi_2' = \frac{c}{a_1 H_1 {q_1}^2} \frac{1}{1+z_2'} \left[q_1 z_2' + (q_1 - 1)\left(\sqrt{1 + 2 q_1 z_2'} - 1 \right) \right].$$

H_0 is the Hubble constant, $a_0 = 1$ is the dimensionless scale factor and q_0 is the deceleration parameter. One can get by without invoking any particular value for H_0 as long as he is willing to report χ_2' in units of h^{-1} Mpc.

The numerical value for the deceleration parameter is required, though, and several assumptions must be made in assigning its value,

$$q_0 \equiv -\left(\frac{\ddot{a} a}{\dot{a}^2} \right)_{t=t_0}.$$

The equation of state, a relationship between pressure P (not to be confused with P in the Liske equations) and energy density ε is usually complicated. In cosmology, which deals primarily with dilute gasses, it takes the relatively simple form of $P = w\varepsilon$ where w is a dimensionless number. The most important values of w in a cosmological sense are those for nonrelativistic gases, relativistic gases (e.g. photons), and dark energy,

$$w_{nonrel} \approx \frac{\langle v^2 \rangle}{3c^2} \ll 1,$$

$$w_{rel} = \frac{1}{3},$$

$$w_\Lambda \approx -1.$$

The WMAP 7 results (Jarosik et al. 2010) calculate $w_\Lambda = -1.12^{+0.42}_{-0.43}$ from WMAP alone and $w_\Lambda = -0.980 \pm 0.053$ if BAO and H_0 data are also included.

Combining the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P),$$

the definition of q_0 and the simple requirement that $\varepsilon = \sum_w \varepsilon_w$ and $P = \sum_w w\varepsilon_w$, one can derive

$$q_0 = \frac{1}{2}\left(\frac{8\pi G}{3c^2 H^2} \right) \sum_w \varepsilon_w (1 + 3w).$$

Further combining with the Friedmann equation in a flat universe,

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t),$$

we find

$$q_0 = \frac{1}{2\varepsilon_c}\sum_w \varepsilon_w(1+3w)$$

$$= \frac{1}{2}\sum_w \Omega_{w,0}(1+3w)$$

$$= \Omega_{r,0} + \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0} = -0.55,$$

where ε_c is the critical energy density of the universe. We assume the universe contains only radiation, matter, and a cosmological constant. With $\Omega_{r,0} = 0$, $\Omega_{m,0} = 0.3$, and $\Omega_{\Lambda,0} = 0.7$, I establish $q_0 = -0.55$.